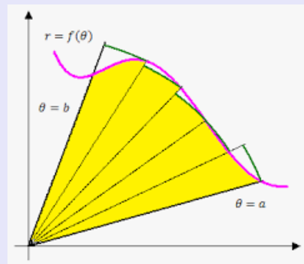
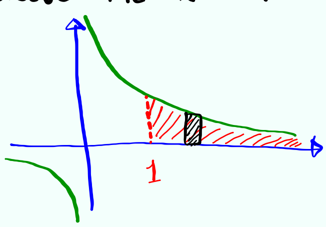
Calculus II

Lecture 12



Feb 19-8:47 AM

Find the area under the curve $f(x)=\frac{1}{x}$, above the x-axis for $x \geq 1$.



$A=\int_1^{\infty} \frac{1}{x}\,dx$ Improper Integral

$=\lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

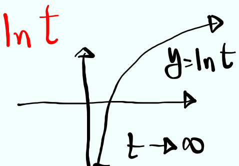
If the limit exists, then the integral is Convergent.

If the limit does not exist, then it is divergent.

$\int_1^t \frac{1}{x}\,dx = \ln x \Big|_1^t = \ln t - \cancel{\ln 1}^{\,0} = \ln t$

$\lim_{t\to\infty} \ln t = \infty$

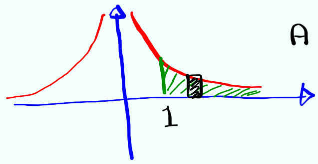
So the integral is divergent.



$t \to \infty$
$y \to \infty$

Jul 1-8:05 AM

Find the area under the curve $f(x) = \frac{1}{x^2}$, above the x-axis for $x \geq 1$.



$$A = \int_1^\infty \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^t x^{-2}\,dx$$

$$\int_1^t x^{-2}\,dx = \frac{x^{-2+1}}{-2+1}\Big|_1^t = \frac{-1}{x}\Big|_1^t = \frac{-1}{t} + \frac{1}{1} = 1 - \frac{1}{t}$$

$$\lim_{t\to\infty} \int_1^t x^{-2}\,dx = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1 - 0 = \boxed{1} \leftarrow \text{Area}$$

Since the limit exists, then we have a convergent improper integral.

Jul 1-8:12 AM

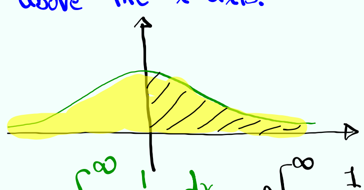
Find the area below $f(x) = \underbrace{\frac{1}{1+x^2}}$ and above the x-axis.

Domain $(-\infty, \infty)$

Even Function

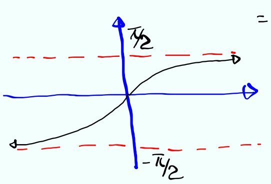
No x-Int $\Rightarrow y \neq 0$

Y-Int $\rightarrow (0, 1)$



$$A = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\int_0^\infty \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} 2\int_0^t \frac{1}{1+x^2}\,dx$$



$$= 2\lim_{t\to\infty} \tan^{-1}x\Big|_0^t$$

$$= 2\lim_{t\to\infty}\left[\tan^{-1}t - \cancel{\tan^{-1}0}^{\,0}\right]$$

$$= 2\cdot\frac{\pi}{2} = \boxed{\pi} \leftarrow \text{Area}$$

$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ is convergent

Jul 1-8:18 AM

Find the area below $f(x)=\frac{1}{\sqrt{x-1}}$, above x-axis between $x=1$ and $x=5$.

$f(x)>0$

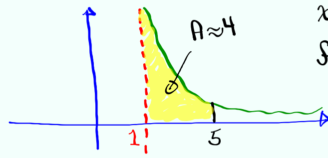
Above x-axis

Domain

$x-1>0$

$x>1$

V.A. $x=1$



$x=1.1$

$f(1.1)=\frac{1}{\sqrt{.1}}$

≈ 3.162

$f(x)$ has discountinuity at $x=1$

$$A=\int_1^5 \frac{1}{\sqrt{x-1}}\,dx$$

$$=\lim_{t\to 1^+}\int_t^5 \frac{1}{\sqrt{x-1}}\,dx$$

$$\int \frac{1}{\sqrt{x-1}}\,dx$$

$u=x-1$

$du=dx$

$$=\int \frac{1}{\sqrt{u}}\,du=\int u^{-1/2}\,du$$

$$=\frac{u^{1/2}}{1/2}+C$$

$$=2\sqrt{x-1}+C$$

$$\int_t^5 \frac{1}{\sqrt{x-1}}\,dx=2\sqrt{x-1}\Big|_t^5$$

$$=2\sqrt{5-1}-2\sqrt{t-1}$$

$$=4-2\sqrt{t-1}$$

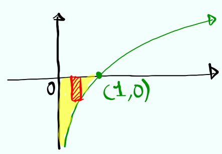
Now $\lim_{t\to 1^+}\left[4-2\sqrt{t-1}\right]$

$$=4-2\sqrt{0}$$

$$=\boxed{4}$$

Jul 1-8:27 AM

Find the area bounded by x-axis, y-axis, and the curve $f(x)=\ln x$.



$$A=\int_0^1 [\,0-\ln x\,]\,dx$$

Top Bottom

we have disc. at $x=0$

$$A=\lim_{t\to 0^+}\int_t^1 -\ln x\,dx=-\lim_{t\to 0^+}\int_t^1 \ln x\,dx=-\lim_{t\to 0^+}\left[-1-t\ln t+t\right]$$

$$\int_t^1 \ln x\,dx=x\ln x-\int x\cdot\frac{1}{x}\,dx=x\ln x-\int 1\,dx$$

$u=\ln x$ $\quad dv=dx$

$du=\frac{1}{x}dx$ $\quad v=x$

$$=x\ln x-x\Big|_t^1$$

$$=\cancel{1\ln 1}^{\,0}-1-t\ln t+t$$

$$=\lim_{t\to 0^+}\left[1+t\ln t-\cancel{t}^{\,0}\right]=1+\lim_{t\to 0^+}t\ln t=0\cdot\infty$$

I.F.

$$\lim_{t\to 0^+}t\ln t=\lim_{t\to 0^+}\frac{\ln t}{\frac{1}{t}}=\lim_{t\to 0^+}\frac{\frac{1}{t}}{\frac{-1}{t^2}}=\lim_{t\to 0^+}(-t)=0$$

$$A=\int_0^1[0-\ln x]\,dx=1$$

Jul 1-8:38 AM

Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ disc. at $x=1$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}dx = \lim_{t\to 1^-} \sin^{-1} t$$

$$\int_0^t \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x \Big|_0^t = \sin^{-1} t - \cancel{\sin^{-1} 0}^{\to 0}$$

$$= \sin^{-1} t$$

$$= \sin^{-1} 1 = \boxed{\frac{\pi}{2}}$$

Jul 1-8:52 AM

Evaluate $\int_0^1 \frac{e^{1/x}}{x^3}dx$ disc. at $x=0$

$$\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}dx = \lim_{t\to 0^+}\int_t^1 \frac{e^{1/x}}{x^3}dx$$

$$\int_t^1 \frac{e^{\frac{1}{x}}}{x^3}dx \qquad u=\frac{1}{x} \qquad du = \frac{-1}{x^2}dx \Rightarrow -x^2du = dx$$

$$\cancel{e^u} + ue^u - \cancel{e^u}$$

$$= \int \frac{e^u}{x\cdot \cancel{x^2}}\cdot -\cancel{x^2}du = -\int ue^u du = -\left[ue^u - e^u\right]$$

$$= e^u - ue^u$$

$$= e^{\frac{1}{x}} - \frac{e^{1/x}}{x}\Big|_t^1$$

$$= (\cancel{e - e})^{\to 0} - \left(e^{\frac{1}{t}} - \frac{e^{\frac{1}{t}}}{t}\right)$$

$$= \frac{e^{1/t}}{t} - e^{\frac{1}{t}}$$

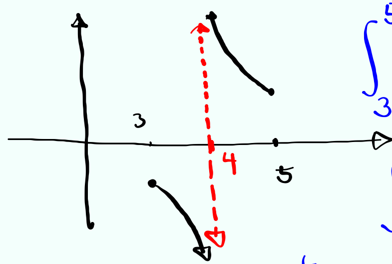
$$\lim_{t\to 0^+}\left[\frac{e^{\frac{1}{t}}}{t} - e^{\frac{1}{t}}\right]$$

$$= \lim_{t\to 0^+} \cancel{e^{\frac{1}{t}}}^{\to\infty}\left[\cancel{\frac{1}{t}}^{\to\infty} - 1\right]$$

$$= \infty \qquad \text{Divergence.}$$

Jul 1-8:57 AM

Evaluate $\displaystyle\int_3^5 \frac{1}{x-4}\,dx$ → disct. at $x=4$



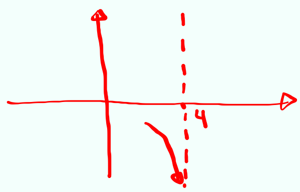
$$\int_3^5 \frac{1}{x-4}\,dx =$$

$$\int_3^{4^-} \frac{1}{x-4}\,dx + \int_{4^+}^5 \frac{1}{x-4}\,dx$$

$$\int_3^{4^-} \frac{1}{x-4}\,dx = \lim_{t\to 4^-} \int_3^t \frac{1}{x-4}\,dx = \lim_{t\to 4^-} \ln|x-4|\Big|_3^t$$

$$= \lim_{t\to 4^-} \ln|t-4| = -\infty$$

Divergent



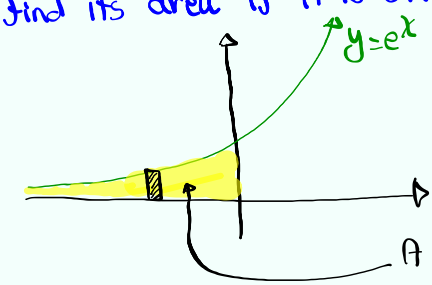
$t = 3.99$

$\ln|3.99-4| \approx -\infty$

Jul 1-9:09 AM

Draw the region $S = \{(x,y) \mid x \le 0,\ 0 \le y \le e^x\}$

Find its area if it is finite.



$$A = \int_{-\infty}^0 e^x\,dx$$

$$A = \lim_{t\to -\infty} \int_t^0 e^x\,dx = \lim_{t\to -\infty} \left[e^x\Big|_t^0 \right]$$

$$= \lim_{t\to -\infty} \left[e^0 - e^t \right]$$

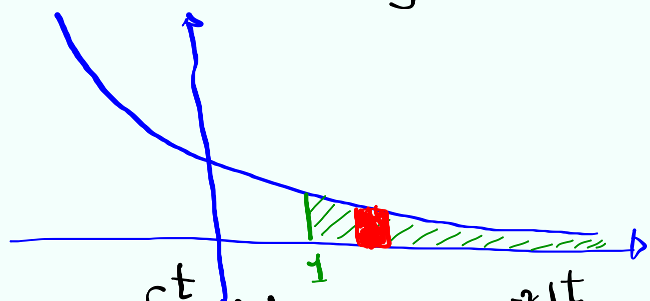
$$= 1 - \cancel{\lim_{t\to -\infty} e^t} \to 0 = \boxed{1}$$

Jul 1-9:17 AM

Draw the region given below

$$S = \{(x,y) \mid x \geq 1 \ , \ 0 \leq y \leq e^{-x}\}$$

Find its area.



$$A = \int_1^{\infty} e^{-x} dx = \lim_{t \to \infty} \int_1^t e^{-x} dx = \lim_{t \to \infty} -e^{-x} \Big|_1^t$$

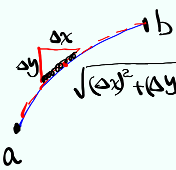
$$= -\lim_{t \to \infty} \left[e^{-t} - e^{-1} \right] = -\lim_{t \to \infty} \left[\frac{1}{e^t} - \frac{1}{e} \right] = \boxed{\frac{1}{e}}$$

Jul 1-9:23 AM

Find the arc length of the curve

$f(x) = 1 + 6x\sqrt{x}$ on $[0,1]$

$$\frac{\Delta y}{\Delta x} \approx f'(x)$$



$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2 \left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]}$$

$$= \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

$f(x) = 1 + 6x^{\frac{3}{2}}$

$f'(x) = 6 \cdot \frac{3}{2} x^{1/2}$

$f'(x) = 9\sqrt{x}$

$1 + [f'(x)]^2 = 1 + 81x$

$$L = \int_0^1 \sqrt{1 + 81x} \, dx$$

$u = 1 + 81x \quad du = 81 dx \quad \frac{du}{81} = dx \quad x = 0 \quad u = 1$

$x = 1 \quad u = 82$

$$L = \int_1^{82} \sqrt{u} \, \frac{du}{81} = \frac{1}{81} \frac{u^{3/2}}{3/2} \Big|_1^{82} = \frac{2}{243} \left[u\sqrt{u} \Big|_1^{82} \right]$$

$$= \boxed{\frac{2}{243} \left[82\sqrt{82} - 1 \right]}$$

Jul 1-9:51 AM

Find the arc length along the curve

$x=\frac{1}{3}\sqrt{y}\,(y-3)$ for $1\le y\le 9$.

$$L=\int_c^d \sqrt{1+[g'(y)]^2}\,dy$$

$g(y)=\frac{1}{3}\sqrt{y}\,(y-3)$

$g(y)=\frac{1}{3}y^{\frac{3}{2}}-y^{1/2}$

$g'(y)=\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2}y^{1/2}-\frac{1}{2}y^{-1/2}$

$g'(y)=\frac{1}{2}\left(\sqrt{y}-\frac{1}{\sqrt{y}}\right)$

$[g'(y)]^2=\frac{1}{4}\left(y-2+\frac{1}{y}\right)$

$$=\int_1^9 \sqrt{1+\frac{1}{4}y-\frac{1}{2}+\frac{1}{4y}}\,dy$$

$$=\int_1^9 \sqrt{\frac{1}{4}y+\frac{1}{2}+\frac{1}{4y}}\,dy$$

$$=\int_1^9 \sqrt{\left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)^2}\,dy=\int_1^9\left(\frac{\sqrt{y}}{2}+\frac{1}{2\sqrt{y}}\right)dy$$

$$=\frac{1}{2}\left[\frac{y^{3/2}}{3/2}+\frac{y^{1/2}}{1/2}\right]\Bigg|_1^9$$

$$=\frac{1}{2}\left[\frac{2}{3}y\sqrt{y}+2\sqrt{y}\right]\Bigg|_1^9=\left[\frac{1}{\cancel{3}}\cdot 9\cdot\cancel{\sqrt{9}}+\sqrt{9}-\frac{1}{3}-1\right]=\boxed{10\tfrac{2}{3}}$$

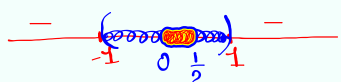
Jul 1-10:01 AM

Find the arc length along the curve

$f(x)=\ln(1-x^2)$ on $\left[0,\frac{1}{2}\right]$.

Domain $1-x^2>0$

$(1+x)(1-x)>0$



$$L=\int_0^{1/2}\sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}\,dx$$

$$=\int_0^{1/2}\sqrt{1+\frac{4x^2}{(1-x^2)^2}}\,dx=\int_0^{1/2}\sqrt{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}}\,dx$$

$$=\int_0^{1/2}\frac{\sqrt{1-2x^2+x^4+4x^2}}{1-x^2}\,dx=\int_0^{1/2}\frac{\sqrt{x^4+2x^2+1}}{1-x^2}\,dx$$

$$=\int_0^{1/2}\frac{\sqrt{(x^2+1)^2}}{1-x^2}\,dx=-\int_0^{1/2}\frac{x^2+1}{x^2-1}\,dx$$

$$\frac{x^2-1\ \ +2}{x^2-1}=\frac{x^2-1}{x^2-1}+\frac{2}{x^2-1}=1+\frac{2}{x^2-1}$$

Jul 1-10:13 AM

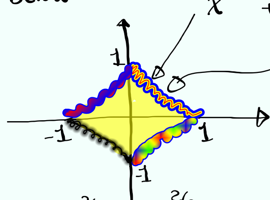
$$-\int_0^{1/2} \frac{x^2+1}{x^2-1}\,dx = -\int_0^{1/2}\left[1+\frac{2}{x^2-1}\right]dx$$

$$= -\left(x + \cancel{2}\cdot\frac{1}{\cancel{2}}\ln\left|\frac{x-1}{x+1}\right|\right)\Bigg|_0^{1/2}$$

$$= -\left[x+\ln\left|\frac{x-1}{x+1}\right|\right]\Bigg|_0^{1/2} = \boxed{}$$

Jul 1-10:23 AM

Find the perimeter of the shaded region below



$x^{2/3}+y^{2/3}=1$

$$L = 4\int_0^1 \sqrt{1+[\quad]^2}\,dx$$

$$x^{2/3}+y^{2/3}=1$$

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx} = \frac{-x^{-1/3}}{y^{-1/3}}$$

$$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{y^{2/3}}{x^{2/3}} = \frac{x^{2/3}}{x^{2/3}}+\frac{y^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}}$$

$$L = 4\int_0^1\sqrt{\frac{1}{x^{2/3}}}\,dx$$

$$= 4\int_0^1 \frac{1}{x^{1/3}}\,dx$$

Improper integral

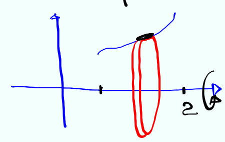
$$\int x^{-1/3}dx = \frac{x^{2/3}}{2/3}\Bigg|_0^1 = \frac{3}{2}$$

$$L = 4\cdot\frac{3}{2} = \boxed{6}$$

Jul 1-10:28 AM

Find the surface area of the solid generated by rotating the arc length along the curve $f(x)=\frac{1}{4}x^2-\frac{1}{2}\ln x$ on $[1,2]$.

about x-axis.



$$S=\int_1^2 2\pi\, f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$S=\int_1^2 2\pi\left(\frac{1}{4}x^2-\frac{1}{2}\ln x\right)\sqrt{1+\left[\frac{x}{2}-\frac{1}{2x}\right]^2}\,dx$$

$$=\int_1^2 2\pi\left[\frac{1}{4}x^2-\frac{1}{2}\ln x\right]\cdot\sqrt{1+\frac{x^2}{4}-\frac{1}{2}+\frac{1}{4x^2}}\,dx$$

$$=\int_1^2 2\pi\left[\frac{1}{4}x^2-\frac{1}{2}\ln x\right]\cdot\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx$$

$$=2\pi\int_1^2\left[\frac{1}{4}x^2-\frac{1}{2}\ln x\right]\cdot\left[\frac{x}{2}+\frac{1}{2x}\right]dx$$

FOIL, Simplify, Do integratin